

Vortex – Kink Interaction and Capillary Waves in a Vector Superfluid

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Abstract

Interaction of a vortex in a circularly polarized superfluid component of a 2d complex vector field with the phase boundary between superfluid phases with opposite signs of polarization leads to a resonant excitation of a “capillary” wave on the boundary. This leads to energy losses by the vortex–image pair that has to cause its eventual annihilation.

Introduction. Several recent publications considered dynamics of a 2d complex vector field, that can be related both to transverse nonlinear optical patterns in polarized light [1, 2, 3] and to the motion of a hypothetical fermionic superfluid [4]. Although dissipation may be non-negligible in optical systems, radiation and dispersion effects play there a primary role. In a scalar case, “acoustic” radiation is the principal mechanism of relaxation to lower energy states. In the vector field, that can be interpreted as a combination of two either miscible or immiscible superfluids, a new relaxation mechanism may appear: radiation of capillary waves. In this communication, we explore the influence of this effect on the interaction of a vortex with an interface between two immiscible superfluids.

Conservative Galilean dynamics of a 2d complex vector field is described by an evolution equation derived from the Hamiltonian \mathcal{H} (see [2]):

$$\mathbf{u}_t = -i \frac{\delta \mathcal{E}}{\delta \mathbf{u}^*}, \quad \mathcal{H} = \int [\nabla \mathbf{u} : \nabla \mathbf{u}^* + V(\mathbf{u})] d^2 \mathbf{x}. \quad (1)$$

The simplest form of a potential that possesses required symmetries to spatial rotations and phase translations but breaks the maximal $SU(2)$ symmetry is

$$V(\mathbf{u}) = \frac{1}{2} \left[(1 - \mathbf{u} \cdot \mathbf{u}^*)^2 + \gamma (\mathbf{u} \cdot \mathbf{u})(\mathbf{u}^* \cdot \mathbf{u}^*) \right], \quad (2)$$

In the representation [2]

$$\mathbf{u} = u_+ \mathbf{U} + u_- \mathbf{U}^*; \quad \mathbf{U} = \frac{1}{\sqrt{2}} \begin{vmatrix} \exp(i\pi/4) \\ \exp(-i\pi/4) \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 1+i \\ 1-i \end{vmatrix}, \quad (3)$$

the dynamic equations take the form

$$-i \partial_t u_{\pm} = \nabla^2 u_{\pm} + \left[1 - |u_{\pm}|^2 - (1 + 2\gamma) |u_{\mp}|^2 \right] u_{\pm}. \quad (4)$$

By setting $u_{\pm} = \rho_{\pm}^{1/2} \exp\left(\frac{i}{2} \phi_{\pm}\right)$, these equations can be brought to a “fluid-mechanical” form:

$$\partial_t \rho_{\pm} + \nabla \cdot (\rho_{\pm} \mathbf{v}_{\pm}) = 0, \quad \partial_t \phi_{\pm} + \frac{1}{2} |\mathbf{v}_{\pm}|^2 + p_{\pm} = 0, \quad (5)$$

where ρ_{\pm} are interpreted as “densities”, $\mathbf{v}_{\pm} = \nabla \theta_{\pm}$, as “velocities” and “pressures” p_{\pm} obey the “equations of state”

$$-p_{\pm} = \frac{1}{2\sqrt{\rho_{\pm}}} \nabla^2 \sqrt{\rho_{\pm}} + 1 - \rho_{\pm} - (1 + 2\gamma) \rho_{\mp}. \quad (6)$$

The two superfluids are “miscible” at $\gamma < 0$, and tend to separate at $\gamma > 0$. In the latter case, two types of topological defects are possible: vortices in either superfluid and kinks separating two oppositely polarized superfluid phases.

The stationary structure of a straight kink is defined by the equations

$$\psi''_{\pm} + \psi_{\pm}[1 - \psi_{\pm}^2 - (1 + 2\gamma)\psi_{\mp}^2] = 0, \quad (7)$$

where $\rho^{\pm} = \psi_{\pm}^2$, and primes denote derivatives with respect to the coordinate y directed normally to the kink. The asymptotic conditions are $\psi_{\pm}(\pm\infty) = \pm 1$, $\psi_{\pm}(\mp\infty) = 0$. The kinks of this kind were studied formerly in a context of dissipative systems [5] and dispersive Kerr media [6]. Recently, the deformation of kinks has been studied numerically in an optical context [7].

Being the only parameter of the model, γ defines both the thickness of the kink and the energy per unit length, or *line tension*

$$\sigma = \int_{-\infty}^{\infty} [(\psi_+'^2 + \psi_-'^2) + \frac{1}{2}(\psi_+^2 + \psi_-^2 - 1)^2 + 2\gamma\psi_+^2\psi_-^2] dy.$$

Multiscale expansion. We shall consider the kink – vortex interaction in the limit $\gamma \gg 1$. This rather technical choice is motivated mainly by simplification of the analysis while retaining results qualitatively valid also in the case $\gamma = O(1)$. Let us note, however, that a rather large value $2\gamma + 1 = 7$ was reported in [8]. Under these conditions, a vortex has a hollow core of unit size, whereas the excess energy of the kink is concentrated in a narrow $O(\gamma^{-1/2})$ layer, and $\sigma = O(\gamma^{1/2}) \gg 1$. Suppose that the kink separates a positively polarized fluid “above” from a negatively polarized fluid “below”, and that a vortex of negative unit charge is placed in the upper fluid at a distance a from the kink. We shall look for a solution in form of an expansion in powers of σ^{-1} . In the zero approximation, the kink is a straight line, taken as the axis $y = 0$. The vortex motion would be determined in this approximation by its interaction with its image, resulting in translation parallel to the straight-line kink with the velocity $c = -1/(2a)$.

In the next approximation, the kink deformation under the action of the pressure field due to the moving vortex can be computed. We shall see that the vortex induces on a kink a capillary wave that takes away the energy of the vortex leading eventually to annihilation of the vortex at the kink. The approach that we use has been applied formerly to a somewhat similar system – interaction of a vortex with a slightly deformable surface of an ideal fluid – by Novikov [9]. There are two principal distinctions between our system and that of Ref. [9]. First, instead of one fluid separated by the interface from an inviscid medium, we have two superfluids with identical physical properties on both sides of the interface. Second, a *capillary* wave is excited in our case, as opposed to a *gravity* wave in Ref. [9].

Since a thin slightly bent kink can be treated as a usual fluid interface possessing a certain line tension σ , its small deformation $h(x) = O(\sigma^{-1})$ is determined by two boundary conditions: the kinematic boundary condition

$$h_t = w^{\pm} - v^{\pm}h_x + hw_y^{\pm}, \quad (8)$$

and the normal stress balance

$$\sigma h_{xx} = p^{(+)} - p^{(-)}, \quad (9)$$

Here x and y are coordinates directed, respectively, along and normally to the unperturbed kink; $v = \phi_x^\pm$ and $w^\pm = \phi_y^\pm$ are the respective velocity components in either fluid computed at the kink line, and $p^{(\pm)}$ are pressures at both sides of the kink that can be computed using the Bernoulli equation (5). The velocities and their derivatives in Eq. (8) are computed at the unperturbed interface position $y = 0$, and the last term in Eq. (8) gives a correction to the vertical velocity at the actual shifted interface.

Inner solution. In the coordinate frame comoving with the vortex, the pressure field is stationary in the leading approximation, and the terms containing the time derivatives in both the Bernoulli equation (5) and the kinematic boundary condition (8) can be neglected. Using the Bernoulli equation rewritten in the moving frame yields the pressure in the “upper” fluid (at $y > 0$) containing the vortex:

$$-p^{(+)} = -c \left(\phi_x^{(+)} + \phi_x^{(-)} \right) + \frac{1}{2} \left| \nabla \phi^{(+)} + \nabla \phi^{(-)} \right|^2 = \frac{a^2 - x^2}{(a^2 + x^2)^2}. \quad (10)$$

Here $\phi^{(\pm)}$ are phase fields (flow potentials) due to, respectively, the vortex at $\{0, a\}$ and its image at $\{0, -a\}$. In the half-plane $y < 0$ there is no motion in the zero approximation, so $p^{(-)} = 0$.

Using Eq. (10) in Eq. (9), and taking the highest elevation $h(0)$ to be equal to zero, readily yields the solution

$$h = -\frac{1}{2\sigma} \ln \left[1 + \left(\frac{x}{a} \right)^2 \right] \quad (11)$$

This solution is valid sufficiently close to the vortex but diverges logarithmically at $x \rightarrow \pm\infty$.

Outer region. At large distances, a different approximation should be used, taking into account non-stationary effects related to excitation of capillary waves propagating along the kink. At a distance $l \gg \sigma \gg a$, the time derivative term in Eq. (8) balances the $O(l^{-1})$ linear term if the characteristic time scale is of $O(l/\sigma)$. The nonlinear terms in Eq. (8) are of $O(\sigma^{-1}l^{-2})$ at these distances, and can be neglected. Thus, the correction to the velocity field can be obtained at large distances by solving the Laplace equation for the $O(\sigma)$ correction $\tilde{\phi}^\pm$ to the flow potential in both fluids, $\nabla^2 \tilde{\phi}^\pm = 0$ with the boundary condition $\tilde{\phi}_y^\pm = h_t$ at $y = 0$.

In the “laboratory” frame we define

$$\hat{\phi}_k^\pm(y, t) = \int_{-\infty}^{\infty} \tilde{\phi}^\pm(x, y, t) e^{-ikx} dx, \quad \hat{h}_k(y, t) = \int_{-\infty}^{\infty} h(x, y, t) e^{-ikx} dx. \quad (12)$$

The solution is

$$\hat{\phi}_k^\pm(y, t) = \mp |k|^{-1} \hat{h}_t e^{-|k|y} \quad (13)$$

The nonlinear term in Eq. (5) can be also neglected under these conditions, so that the Fourier components \hat{p}_k^\pm of the additional pressure in both fluids are immediately computed as

$$\hat{p}_k^\pm(y, t) = \pm |k|^{-1} \hat{h}_{tt} e^{-|k|y} \quad (14)$$

The solution is closed using the Fourier transform of the normal stress boundary condition (9)

$$-\sigma k^2 \hat{h} = 2|k|^{-1} \hat{h}_{tt} + \hat{p}^{(0)} e^{-ikct}, \quad \hat{p}^{(0)} = \frac{|k|}{4\pi} e^{-|k|a}. \quad (15)$$

where $\hat{p}^{(0)}$ is the Fourier transform of the zero-order pressure in the upper fluid given by Eq. (10).

It is convenient to rewrite Eq. (15) in the form

$$\hat{h}_{tt} + \omega^2 \hat{h} = f(k) e^{-ikct}, \quad (16)$$

where $f(k) = -(k^2/8\pi) e^{-|k|a}$, $\omega^2 = \frac{1}{2} \sigma |k|^3$. The forcing term in the right-hand side is *resonant* at $\omega = kc$. This resonance gives the most important contribution to the surface deformation that would not decay at large distances and long times. In order to obtain the asymptotic form of the resonant wave, we introduce, following Novikov [9], a modified forcing $f(k) e^{(-ikc+\epsilon)t}$, and take subsequently the limit $\epsilon \rightarrow 0$. Solving Eq. (16) with the modified forcing yields

$$h = \frac{e^{\epsilon t}}{2\pi} \int_{-\infty}^{\infty} \frac{f(k) e^{ik(x-ct)} dk}{\omega^2 - (kc + i\epsilon)^2}. \quad (17)$$

The integral is computed by closing a contour in the complex k -plane. The asymptotic solution as $|x| \rightarrow \infty$ is

$$h = \begin{cases} \frac{1}{\sigma} \sin \frac{x-ct}{2\sigma a^2} & \text{at } x < 0 \\ 0 & \text{at } x > 0 \end{cases}. \quad (18)$$

Obviously, the wave number of the capillary wave generated by the vortex motion $K = 1/(2a^2\sigma)$ satisfies the resonance condition $\omega(K) = |c|K$. It is remarkable that, in contrast to the case of gravity waves considered by Novikov [9], the wave runs *ahead* of the vortex. This circumstance is connected with the fact that the group velocity of capillary waves exceeds their phase velocity.

The energy flux connected with the wave (18) that diminishes the energy of the vortex/image pair is [10]

$$F = C(K) E_w, \quad (19)$$

where E_w is the energy density and $C(K) = 3/4a$ is the group velocity of waves. The full (surface) energy density is the sum of the deformation energy E_1 of the kink and the

kinetic energy of both fluids E_2 . The calculations show that

$$F = \frac{9}{32\sigma^3 a^5}. \quad (20)$$

Because the energy of the vortex–image pair is

$$E_p = \frac{1}{4\pi} \ln \frac{2a}{\delta}, \quad (21)$$

where δ is the radius of the vortex core, we find that the balance condition $dE_p/dt = -F$ leads to the following dynamics of the vortex:

$$\frac{da}{dt} = -\frac{9\pi}{8\sigma^3} \frac{1}{a^4}. \quad (22)$$

Thus, the vortex–image pair tends to annihilate, and the distance between the vortex and the image is governed by the law

$$a \sim (t_c - t)^{1/5}. \quad (23)$$

Certainly, this law is valid while a is still large compared to the size of the vortex core.

Conclusion. We found that the interaction of the vortex in a circularly polarized component with the phase boundary between phases with opposite signs of polarization generates a specific capillary wave on the boundary. This phenomenon leads to energy losses by the vortex–image pair that has to terminate in its eventual annihilation.

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